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SHOCK AND ACCELERATION WAVES WITH LARGE AMPLITUDES IN LAMINATED COMPOSITE PLATES

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Abstract—Propagation of shock and acceleration waves with large amplitudes is studied. The geometrical nonlinearity in the von Karman sense is included in deriving the plate equations. The dynamical conditions on the wave fronts are derived from the three-dimensional conditions in a way consistent with the derivation of the plate equations. General equations governing the propagation velocities are obtained. Solutions are presented for the case where the plates are initially at rest. It is found that, in this case, the large amplitude has a substantial effect only on the transverse shear shock wave. Finally, stability of the wave front is discussed.

1. INTRODUCTION

Due to their particular in-plane reinforcement, laminated composite plates are usually stiff in in-plane extension and bending, while transverse shear deformations are not accompanied by the comparable rigidity. As a consequence, it is very likely that, if the plate is subjected to an intensive transverse loading such as a lateral impact, large deflections could result due to the relatively weak shear rigidity. The wave fronts generated by such impulsive loadings could then propagate under the influence of large amplitudes.

Work in the study of wave front propagation has been limited to the use of linear plate theories. Moon[1] presented an analysis of wave surfaces in an equivalent homogeneous orthotropic plate for laminated plates. The acceleration wave was investigated. Sun[2] studied shock fronts in laminated plates by using the ray theory. The propagation velocity, the expression for the location of the wave front, and the ray geometry were presented.

In this paper, propagation of shock and acceleration waves with large amplitudes is studied. The geometrical nonlinearity in the von Karman sense is included in deriving the plate equations which can also account for the transverse shear deformation. The dynamical conditions on the wave fronts are derived from the three-dimensional conditions in a way that is consistent with the derivation of the plate equations. Solutions are presented for the case where plates are assumed to be initially undisturbed. In a second paper to follow the effect of initial deformations on the propagation of these wave fronts will be discussed.

2. SHOCK WAVE

Consider a laminated plate of fiber-reinforced composite materials. Let x, y, and z be the Lagrangian coordinates, and the x-y plane coincide with the mid-plane of the plate with the z-axis perpendicular to it.

In the Lagrangian description, the equation for conservation of linear momentum at the shock front can be expressed in the form[3]

$$[L_{ij}]n_j = -\rho_0 c_n [\dot{\bar{u}}_i] \tag{1}$$

where L_{ij} is the Lagrangian stress tensor, n_j is the unit normal vector to the shock surface, ρ_0 is the initial mass density, $\bar{u}_i$ is the displacement vector, c_n is the normal component of the shock propagation velocity, and a dot indicates the time derivative. In eqn (1), the jump operator is defined as

$$[f] = f_b - f_a \tag{2}$$

where subscripts a and b refer to the values of the function f immediately ahead and behind the wave front, respectively.

The dynamical conditions on the wave front given by equation (1) can be written in terms of the Kirchhoff stress tensor S_{ij} as

$$\left[\left(\delta_{ik}+\frac{\partial \bar{u}_j}{\partial x_i}\right)S_{ik}\right]n_i=-\rho_0 c_n[\dot{\bar{u}}_i] \qquad (3)$$

where δ_{jk} is the Kronecker delta.

The kinematical condition of compatibility is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}[f]=\left[\frac{\partial f}{\partial t}\right]+c_i\left[\frac{\partial f}{\partial x_i}\right] \qquad (4a)$$

where c_i are components of the Lagrangian velocity of the wave front. If $[f]=0$, eqn (4a) reduces to[2]

$$\left[\frac{\partial f}{\partial x_i}\right]=-\frac{n_i}{c_n}\left[\frac{\partial f}{\partial t}\right]. \qquad (4b)$$

We are concerned with shock waves propagating in the plane of the plate so that the wave front depends only on the two spatial variables x and y. Thus, we have $n_z=0$ in the analysis to follow. The displacement field in the plate will be approximated by

$$\begin{aligned}\bar{u}_x&=u(x,y,t)+z\psi_x(x,y,t)\\ \bar{u}_y&=v(x,y,t)+z\psi_y(x,y,t)\\ \bar{u}_z&=w(x,y,t)\end{aligned} \qquad (5)$$

where u, v and w represent the displacements in the mid-plane of the plate, and ψ_x and ψ_y represent rotations of the cross-sections.

Substituting eqn (5) in (3) and retaining the non-linear terms that involve large slopes $\partial w/\partial x$ and $\partial w/\partial y$, we obtain the dynamical conditions consistent with the von Karman large deflection theory of plates as

$$\begin{aligned}&[S_{xx}]n_x+[S_{xy}]n_y=-\rho_0 c_n([\dot{u}]+z[\dot{\psi}_x])\\ &[S_{xy}]n_x+[S_{yy}]n_y=-\rho_0 c_n([\dot{v}]+z[\dot{\psi}_y])\\ &\left[S_{xz}+\frac{\partial w}{\partial x}S_{xx}+\frac{\partial w}{\partial y}S_{xy}\right]n_x+\left[S_{yz}+\frac{\partial w}{\partial x}S_{xy}+\frac{\partial w}{\partial y}S_{yy}\right]n_y=-\rho_0 c_n[\dot{w}].\end{aligned} \qquad (6)$$

It should be noted that in deriving eqn (6) the condition $n_z=0$ has been used.

Integrating eqn (6) and the first two equations multiplied by z over the thickness of plate h, we obtain

$$\begin{aligned}&[N_x]n_x+[N_{xy}]n_y=-c_nP[\dot{u}]-c_nR[\dot{\psi}_x]\\ &[N_{xy}]n_x+[N_y]n_y=-c_nP[\dot{v}]-c_nR[\dot{\psi}_y]\\ &[M_x]n_x+[M_{xy}]n_y=-c_nR[\dot{u}]-c_nI[\dot{\psi}_x]\\ &[M_{xy}]n_x+[M_y]n_y=-c_nR[\dot{v}]-c_nI[\dot{\psi}_y]\end{aligned} \qquad (7)$$

and

$$[Q_x]n_x+[Q_y]n_y+\left[\frac{\partial w}{\partial x}N_x+\frac{\partial w}{\partial y}N_{xy}\right]n_x+\left[\frac{\partial w}{x}N_{xy}+\frac{\partial w}{\partial y}N_y\right]n_y=-c_nP[\dot{w}] \qquad (8)$$

where

$$(N_x,N_y,N_{xy})=\int_{-h/2}^{h/2}(S_{xx},S_{yy},S_{xy})\,\mathrm{d}z$$

$$(M_x, M_y, M_{xy}) = \int_{-h/2}^{h/2} (S_{xx}, S_{yy}, S_{xy}) z \, dz$$

$$(Q_x, Q_y) = \int_{-h/2}^{h/2} (S_{xz}, S_{yz}) \, dz$$

$$(P, R, I) = \int_{-h/2}^{h/2} \rho_0(1, z, z^2) \, dz. \tag{9}$$

We now consider a laminated plate consisting of a finite number of layers of fiber-reinforced materials. Each layer has a set of reduced stiffness coefficients Q_{ij} [4]. Since we have assumed that the nonlinearity was geometrical in nature, we will employ the usual linear stress-strain relations given by

$$\begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{xy} \end{Bmatrix} = \begin{Bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{Bmatrix} \begin{Bmatrix} E_{xx} \\ E_{yy} \\ 2E_{xy} \end{Bmatrix} \tag{10}$$

and

$$\begin{Bmatrix} S_{xz} \\ S_{yz} \end{Bmatrix} = \begin{Bmatrix} Q_{55} & Q_{45} \\ Q_{45} & Q_{44} \end{Bmatrix} \begin{Bmatrix} 2E_{xz} \\ 2E_{yz} \end{Bmatrix} \tag{11}$$

where $E_{xx}, E_{yy}, \ldots$ etc. are the Lagrangian strain components. Retaining only the nonlinear terms involving $\partial w/\partial x$ and $\partial w/\partial y$ and using eqn (5), the Lagrangian strain components assume the following form:

$$\begin{aligned}
E_{xx} &= \frac{\partial u}{\partial x} + z\frac{\partial \psi_x}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\
E_{yy} &= \frac{\partial v}{\partial y} + z\frac{\partial \psi_y}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\
2E_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + z\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right) + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \\
2E_{xz} &= \psi_x + \frac{\partial w}{\partial x} \\
2E_{yz} &= \psi_y + \frac{\partial w}{\partial y}.
\end{aligned} \tag{12}$$

Substituting eqn (12) in eqns (10) and (11) and then in (9) we obtain

$$\{N\} = \{A\} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} + \{B\} \begin{Bmatrix} \dfrac{\partial \psi_x}{\partial x} \\ \dfrac{\partial \psi_y}{\partial y} \\ \dfrac{\partial \psi_x}{\partial y} + \dfrac{\partial \psi_y}{\partial x} \end{Bmatrix} + \{N_n\}$$

$$\{M\} = \{B\} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} + \{D\} \begin{Bmatrix} \dfrac{\partial \psi_x}{\partial x} \\ \dfrac{\partial \psi_y}{\partial y} \\ \dfrac{\partial \psi_x}{\partial y} + \dfrac{\partial \psi_y}{\partial x} \end{Bmatrix} + \{M_n\} \tag{13}$$

and

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \begin{Bmatrix} A_{55} & A_{45} \\ A_{45} & A_{44} \end{Bmatrix} \begin{Bmatrix} \psi_x + \dfrac{\partial w}{\partial x} \\ \psi_y + \dfrac{\partial w}{\partial y} \end{Bmatrix} \tag{14}$$

where

$$\{N\} = \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix}, \quad \{M\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} \tag{15}$$

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} Q_{ij}(1, z, z^2)\,\mathrm{d}z \qquad i,j = 1,2,6 \tag{16a}$$

$$A_{ij} = \int_{-h/2}^{h/2} Q_{ij}\,\mathrm{d}z \qquad i,j = 4,5 \tag{16b}$$

and

$$(\{N_n\}, \{M_n\}) = (\{A\}, \{B\}) \begin{Bmatrix} \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2 \\ \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}\right)^2 \\ \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} \end{Bmatrix}. \tag{17}$$

For shock waves, the plate displacement components remain continuous across the wave front, i.e.

$$[u_i] = 0 \quad i = 1\text{–}5 \tag{18}$$

where $u_1 = u$, $u_2 = v$, $u_3 = \psi_x$, $u_4 = \psi_y$, $u_5 = w$. From the kinematic condition of compatibility, eqn (4b), we have

$$\left[\frac{\partial u_j}{\partial x_i}\right] = -\frac{n_i}{c_n}[\dot{u}_j] \quad i = 1,2, \quad j = 1\text{–}5. \tag{19}$$

Using the relations given by eqn (19) together with eqns (13) and (14), we can obtain the following relations from eqns (7) and (8):

$$(\{A^*\} - c_n^2 P\{I\})\begin{Bmatrix} [\dot{u}] \\ [\dot{v}] \end{Bmatrix} + (\{B^*\} - c_n^2 R\{I\})\begin{Bmatrix} [\dot{\psi}_x] \\ [\dot{\psi}_y] \end{Bmatrix} = c_n\{T\}^T\{[N_n]\} \tag{20a}$$

$$(\{B^*\} - c_n^2 R\{I\})\begin{Bmatrix} [\dot{u}] \\ [\dot{v}] \end{Bmatrix} + (\{D^*\} - c_n^2 I\{I\})\begin{Bmatrix} [\dot{\psi}_x] \\ [\dot{\psi}_y] \end{Bmatrix} = c_n\{T\}^T\{[M_n]\} \tag{20b}$$

$$-c_n[k_n] + (k_5 - c_n^2 P)[\dot{w}] = 0 \tag{20c}$$

where $\{I\}$ is a 2×2 unity matrix,

$$\underset{2\times 2}{(\{A^*\},}\ \underset{2\times 2}{\{B^*\},}\ \underset{2\times 2}{\{D^*\})} = \{T\}^T\ \underset{3\times 3}{(\{A\},}\ \underset{3\times 3}{\{B\},}\ \underset{3\times 3}{\{D\})}\ \underset{3\times 2}{\{T\}} \tag{21}$$

$$\{T\} = \begin{Bmatrix} n_x & 0 \\ 0 & n_y \\ n_y & n_x \end{Bmatrix} \tag{22}$$

$$k_5 = n_x^2 A_{55} + n_y^2 A_{44} + 2 n_x n_y A_{45} \tag{23}$$

and

$$[k_n] = \left[\frac{\partial w}{\partial x} N_x + \frac{\partial w}{\partial y} N_{xy}\right] n_x + \left[\frac{\partial w}{\partial x} N_{xy} + \frac{\partial w}{\partial y} N_y\right] n_y. \tag{24}$$

It now remains to find the expressions for the nonlinear terms $\{[N_n]\}$, $\{[M_n]\}$ and $[k_n]$ in terms of the jumps of the time derivatives $[\dot{u}]$, $[\dot{v}]\ldots$, etc.

Consider two functions G and H. Applying the jump operator (2) on GH, we can easily show that

$$[GH] = G_b H_b - G_a H_a = H_a[G] + G_a[H] + [G][H]. \tag{25}$$

Applying the above jump relation on the nonlinear terms in eqns (17) and (24), we obtain (see Appendix)

$$\begin{aligned}
\{[N_n]\} &= \frac{1}{2}\frac{[\dot{w}]^2}{c_n^2}\{A\}\{T\}\{n\} - \frac{[\dot{w}]}{c_n}\{A\}\{T\}\{E_a\} \\
\{[M_n]\} &= \frac{1}{2}\frac{[\dot{w}]^2}{c_n^2}\{B\}\{T\}\{n\} - \frac{[\dot{w}]}{c_n}\{B\}\{T\}\{E_a\} \\
[k_n] &= \frac{[\dot{w}]}{c_n^2}\{n\}^T\{A^*\}\begin{Bmatrix}[\dot{u}]\\ [\dot{v}]\end{Bmatrix} - \frac{1}{c_n}\{E_a\}^T\{A^*\}\begin{Bmatrix}[\dot{u}]\\ [\dot{v}]\end{Bmatrix} \\
&\quad + \frac{[\dot{w}]}{c_n^2}\{n\}^T\{B^*\}\begin{Bmatrix}[\dot{\psi}_x]\\ [\dot{\psi}_y]\end{Bmatrix} - \frac{1}{c_n}\{E_a\}^T\{B^*\}\begin{Bmatrix}[\dot{\psi}_x]\\ [\dot{\psi}_y]\end{Bmatrix} \\
&\quad - \frac{[\dot{w}]}{c_n}\Big(\{N_a\}^T\{T\}\{n\} + \{E_a\}^T\{A^*\}\{E_a\} - \frac{3}{2}\frac{[\dot{w}]}{c_n}\{E_a\}^T\{A^*\}\{n\} \\
&\quad + \frac{1}{2}\frac{[\dot{w}]^2}{c_n^2}\{n\}^T\{A^*\}\{n\}\Big)
\end{aligned} \tag{26}$$

where

$$\{n\} = \begin{Bmatrix} n_x \\ n_y \end{Bmatrix}, \{E\} = \begin{Bmatrix} \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial y} \end{Bmatrix}. \tag{27}$$

The quantities $\{N_a\}$ and $\{E_a\}$ are the values of $\{N\}$ and $\{E\}$, respectively, evaluated at points immediately ahead of the wave front. It should be noted that the expressions given by eqn (26) are very general and can be used to study cases such as large amplitude shock wave propagation in an initially deformed plate.

Substituting eqn (26) in eqn (20), we finally arrive at the following system of equations:

$$\{a_{ij}\}\{[\dot{u}_j]\} = \{0\} \qquad i,j = 1,2,3,4,5 \tag{28}$$

where

$$\{a_{ij}\} = \begin{Bmatrix}
A^*_{11} - c_n^2 P & A^*_{12} & B^*_{11} - c_n^2 R & B^*_{12} & k_1{}^0 - \frac{1}{2}k_1\frac{[\dot{w}]}{c_n} \\
A^*_{21} & A^*_{22} - c_n^2 P & B^*_{21} & B^*_{22} - c_n^2 R & k_2{}^0 - \frac{1}{2}k_2\frac{[\dot{w}]}{c_n} \\
B^*_{11} - c_n^2 R & B^*_{12} & D^*_{11} - c_n^2 I & D^*_{12} & k_3{}^0 - \frac{1}{2}k_3\frac{[\dot{w}]}{c_n} \\
B^*_{21} & B^*_{22} - c_n^2 R & D^*_{21} & D^*_{22} - c_n^2 I & k_4{}^0 - \frac{1}{2}k_4\frac{[\dot{w}]}{c_n} \\
k_1{}^0 - k_1\frac{[\dot{w}]}{c_n} & k_2{}^0 - k_2\frac{[\dot{w}]}{c_n} & k_3{}^0 - k_3\frac{[\dot{w}]}{c_n} & k_4{}^0 - k_4\frac{[\dot{w}]}{c_n} & k_5 - c_n^2 P + \delta_n
\end{Bmatrix} \tag{29}$$

$$\{k\} = \begin{Bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{Bmatrix} = \begin{Bmatrix} \{A^*\} \\ \{B^*\} \end{Bmatrix} \{n\} \tag{30a}$$

$$\{k^0\} = \begin{Bmatrix} k_1^0 \\ k_2^0 \\ k_3^0 \\ k_4^0 \end{Bmatrix} = \begin{Bmatrix} \{A^*\} \\ \{B^*\} \end{Bmatrix} \{E_a\} \tag{30b}$$

and

$$\delta_n = \gamma^0 - \frac{3}{2}\frac{[\dot{w}]}{c_n}\gamma_1^0 + \frac{1}{2}\frac{[\dot{w}]^2}{c_n^2}\gamma \tag{31a}$$

where

$$\begin{aligned} \gamma^0 &= \{E_a\}^T\{A^*\}\{E_a\} + \{n\}^T\{T\}^T\{N_a\} \\ \gamma_1^0 &= \{E_a\}^T\{A^*\}\{n\} \\ \gamma &= \{n\}^T\{A^*\}\{n\}. \end{aligned} \tag{31b}$$

In order that a nontrivial solution exists for eqn (28) we must have

$$|a_{ij}| = 0. \tag{32}$$

From this equation, the propagation velocity can be determined for a shock front with a specified normal vector, and a given amplitude $[\dot{w}]$ at the wave front. The values of $\{E_a\}$ and $\{N_a\}$ which depend on the previous disturbances in the plate before the present shock arrives have to be predetermined also.

From the definitions of k_3, k_4, k_3^0 and k_4^0 given by eqn (30), we can see that for symmetric laminates ($B_{ij} = 0$) all these quantities vanish, and that from eqn (29) the bending modes are decoupled from the other modes. Therefore, the bending modes are affected by the geometrical nonlinearity only through the coupling terms B_{ij}.

The linear solution given by [2] can be obtained by setting $k_i = k_i^0 = 0$ ($i = 1,2,3,4$) and $\delta_n = 0$.

3. UNDISTURBED MEDIA

In this paper, we will consider the case of initially undisturbed laminated plates. In a second paper to follow, the effect of initial deformations will be presented in details. For a plate that is initially at rest, we have

$$\{N_a\} = \{E_a\} = \{0\}. \tag{33}$$

It is assumed that all the lamina of the plate have the same mass density. As a consequence,

$$R = 0. \tag{34}$$

By using equations (34) and (33), equation (32) can be written as

$$\begin{vmatrix} A^*_{11} - c_n^2 P & A^*_{12} & B^*_{11} & B^*_{12} & k_1[\dot{w}] \\ A^*_{21} & A^*_{22} - c_n^2 P & B^*_{21} & B^*_{22} & k_2[\dot{w}] \\ B^*_{11} & B^*_{12} & D^*_{11} - c_n^2 I & D^*_{12} & k_3[\dot{w}] \\ B^*_{21} & B^*_{22} & D^*_{21} & D^*_{22} - c_n^2 I & k_4[\dot{w}] \\ k_1[\dot{w}] & k_2[\dot{w}] & k_3[\dot{w}] & k_4[\dot{w}] & 2c_n^2(k_5 - c_n^2 P) + \gamma[\dot{w}]^2 \end{vmatrix} = 0 \tag{35}$$

It may appear from eqn (35) that there would be six roots for c_n^2. However, it can be shown that one of the roots is trivial, and thus, the order of the expanded polynomial in c_n^2 can be reduced by one. This can be done as follows: Multiplying the first row by $-n_1[\dot{w}]$ and the second row by $-n_y[\dot{w}]$ and adding them to the 5th row together with

$$\gamma = n_x k_1 + n_y k_2 \tag{36}$$

we obtain from eqn (35)

$$\begin{vmatrix} A^*_{11} - c_n^2 P & A^*_{12} & B^*_{11} & B^*_{12} & k_1[\dot{w}] \\ A^*_{21} & A^*_{22} - c_n^2 P & B^*_{21} & B^*_{22} & k_2[\dot{w}] \\ B^*_{11} & B^*_{12} & D^*_{11} - c_n^2 I & D^*_{12} & k_3[\dot{w}] \\ B^*_{21} & B^*_{22} & D^*_{21} & D^*_{22} - c_n^2 I & k_4[\dot{w}] \\ n_x P[\dot{w}] & n_y P[\dot{w}] & 0 & 0 & 2(k_5 - c_n^2 P) \end{vmatrix} = 0. \tag{37}$$

The above equation will yield five roots only for c_n^2.

It may also seem that we can increase the value of $[\dot{w}]$ as much as we please. However, according to this analysis there is a limiting value for $[\dot{w}]$ at which the root corresponding to the transverse shear wave front becomes zero. When $[\dot{w}]$ increases beyond this value the root becomes negative, and, as a result, the value of c_n becomes imaginary which is not physically meaningful. The limiting value of $[\dot{w}]$ can be obtained analytically from equation (37) as follows: The constant term in the polynomial expansion in c_n^2 of equation (37) is obtained by setting $c_n = 0$. In the resulting determinant if we multiply the first column by $-n_x[\dot{w}]$ and the second column by $-n_y[\dot{w}]$ and add them to the 5th column, then the constant term is obtained as

$$(2k_5 - [\dot{w}]^2)\begin{vmatrix} \{A^*\} & \{B^*\} \\ \{B^*\} & \{D^*\} \end{vmatrix} \tag{38}$$

which is equal to zero if $[\dot{w}]^2 = 2k_5$, and negative if $[\dot{w}]^2$ is greater than $2k_5$. This would render one of the roots of c_n^2 negative[5]. Therefore, the following inequality must be satisfied at all directions,

$$[\dot{w}]^2 \leq 2(n_x^2 A_{55} + n_y^2 A_{44} + 2n_x n_y A_{45}). \tag{39}$$

The above relation tells us that, according to the present analysis, the jump in the particle velocity $[\dot{w}]$ should be less or equal to the shear wave velocity in the linear case multiplied by square root of 2 in order that a transverse shear wave front can propagate.

4. ACCELERATION WAVE

In this section, we will consider the second order wave front (or acceleration wave). The analysis of higher order wave fronts follows exactly the same path as described in the previous section.

For the acceleration wave, besides the condition given by eqn (18), we also require the continuity of particle velocities, the derivatives of the plate displacements and the stress and moment resultants at the wave front. Using the kinematic condition of compatibility given by eqn (4b), we obtain in each layer

$$\left[\frac{\partial L_{ij}}{\partial x_k}\right] = -\frac{n_k}{c_n}[\dot{L}_{ij}] \tag{40}$$

and

$$\left[\frac{\partial \dot{u}_i}{\partial x_k}\right] = -\frac{n_k}{c_n}[\ddot{u}_i]. \tag{41}$$

The dynamical condition at the wave front is now obtained directly from the equations of motion as

$$\left[\frac{\partial L_{ij}}{\partial x_k}\right] = \rho_0[\ddot{u}_i]. \tag{42}$$

Combining eqns (40) and (42) we obtain

$$[\dot{L}_{ni}]n_i = -\rho_0 c_n [\ddot{u}_i]. \tag{43}$$

Again we replace in the above equations the Lagrangian stress tensor by the Kirchoff stress tensor using the relation

$$L_{ni} = \left(\delta_{ik} + \frac{\partial \bar{u}_i}{\partial X_k}\right) S_{ik}. \tag{44}$$

Using the approximate plate displacement field given by eqn (5) in eqns (41) and (43) and making the appropriate integration of eqn (43) as before, we finally arrive at the following relations similar to eqns (7) and (8):

$$\begin{aligned}
\{T\}^T\{[\dot{N}]\} &= -c_n P\begin{Bmatrix}[\ddot{u}]\\ [\ddot{v}]\end{Bmatrix} - c_n R\begin{Bmatrix}[\ddot{\psi}_x]\\ [\ddot{\psi}_y]\end{Bmatrix}\\
\{T\}^T\{[\dot{M}]\} &= -c_n R\begin{Bmatrix}[\ddot{u}]\\ [\ddot{v}]\end{Bmatrix} - c_n I\begin{Bmatrix}[\ddot{\psi}_x]\\ [\ddot{\psi}_y]\end{Bmatrix}\\
[\dot{Q}_x]n_x + [\dot{Q}_y]n_y + [\dot{k}_n] &= -c_n P[\ddot{w}]
\end{aligned} \tag{45}$$

in which

$$\begin{aligned}
\{[\dot{N}]\} &= -\frac{1}{c_n}\{A\}\{T\}\begin{Bmatrix}[\ddot{u}]\\ [\ddot{v}]\end{Bmatrix} - \frac{1}{c_n}\{B\}\{T\}\begin{Bmatrix}[\ddot{\psi}_x]\\ [\ddot{\psi}_y]\end{Bmatrix} + \{[\dot{N}_n]\}\\
\{[\dot{M}]\} &= -\frac{1}{c_n}\{B\}\{T\}\begin{Bmatrix}[\ddot{u}]\\ [\ddot{v}]\end{Bmatrix} - \frac{1}{c_n}\{D\}\{T\}\begin{Bmatrix}[\ddot{\psi}_x]\\ [\ddot{\psi}_y]\end{Bmatrix} + \{[\dot{M}_n]\}\\
\begin{Bmatrix}[\dot{Q}_x]\\ [\dot{Q}_y]\end{Bmatrix} &= -\frac{[\ddot{w}]}{c_n}\begin{bmatrix}A_{55} & A_{45}\\ A_{45} & A_{44}\end{bmatrix}
\end{aligned} \tag{46}$$

where $\{[\dot{N}_n]\}$, $\{[\dot{M}_n]\}$ and $[\dot{k}_n]$ are the nonlinear terms which can be obtained by differentiating eqns (17) and (24) with the operator given by eqn (2). To do that, consider again two functions G and H. We have

$$\frac{\partial}{\partial t}(GH) = G\dot{H} + H\dot{G}. \tag{47}$$

Applying the relation given by eqn (25) to the above quantity we obtain

$$\left[\frac{\partial}{\partial t}(GH)\right] = G_a[\dot{H}] + H_a[\dot{G}] + \dot{H}_a[G] + \dot{G}_a[H] + [G][\dot{H}] + [H][\dot{G}]. \tag{48}$$

If G and H are constant across the wave front, then eqn (48) reduces to

$$\left[\frac{\partial}{\partial t}(GH)\right] = G_a[\dot{H}] + H_a[\dot{G}]. \tag{49}$$

By employing now the operator given by eqn (49), eqns (17) and (24), after differentiation, can be written as

$$\begin{aligned}
\{[\dot{N}_n]\} &= -\frac{[\ddot{w}]}{c_n}\{A\}\{T\}\{E_a\}\\
\{[\dot{M}_n]\} &= -\frac{[\ddot{w}]}{c_n}\{B\}\{T\}\{E_a\}
\end{aligned} \tag{50}$$

$$[\dot{k}_n] = -\frac{1}{c_n}\{E_a\}^T\{A^*\}\begin{Bmatrix}[\ddot{u}]\\ [\ddot{v}]\end{Bmatrix} - \frac{1}{c_n}\{E_a\}^T\{B^*\}\begin{Bmatrix}[\ddot{\psi}_x]\\ [\ddot{\psi}_y]\end{Bmatrix} - \frac{[\ddot{w}]}{c_n}(\{E_a\}^T\{A^*\}\{E_a\} + \{n\}^T\{T\}^T\{N_a\}).$$

Introducing eqns (50) and (46) into eqn (45), we obtain the eigenvalue problem

$$\{b_{ij}\}\{[\ddot{u}_j]\} = \{0\} \tag{51}$$

where the matrix $\{b_{ij}\}$ is similar to $\{a_{ij}\}$ and can be derived from the general eqn (29) by putting $k_i = 0$ ($i = 1,2,3,4$), $\delta_n = \gamma_0$ and the remaining parameters k_i^0, γ_0 and $\{E_a\}$ can be obtained from eqns (30b), (31) and (27), respectively. The explicit expression of $\{b_{ij}\}$ is given by

$$\{b_{ij}\} = \begin{Bmatrix} A^*_{11} - c_n^2 P & A^*_{12} & B^*_{11} & B^*_{12} & k_1^0 \\ A^*_{21} & A^*_{22} - c_n^2 P & B^*_{21} & B^*_{22} & k_2^0 \\ B^*_{11} & B^*_{12} & D^*_{11} - c_n^2 I & D^*_{12} & k_3^0 \\ B^*_{21} & B^*_{22} & D^*_{21} & D^*_{22} - c_n^2 I & k_4^0 \\ k_1^0 & k_2^0 & k_3^0 & k_4^0 & k_5 - c_n^2 P + \gamma^0 \end{Bmatrix}. \tag{52}$$

For an undisturbed medium immediately ahead of the wave front we have $\{N_a\} = \{E_a\} = \{0\}$. Hence, from (30b) and (31b) we obtain

$$k_i^0 = 0 \qquad i = 1,2,3,4$$
$$\gamma_0 = 0 \tag{53}$$

Substituting eqn (53) in (52), we find that the resulting matrix $\{b_{ij}\}$ is identical to the matrix $\{a_{ij}\}$ for shocks at infinitesimal amplitudes. In other words, the geometrical nonlinearity does not affect the propagation of the acceleration if it is not present ahead of the wave front.

5. NUMERICAL RESULTS

We consider the laminates consisting of layers of a typical graphite-epoxy composite. A 0-layer indicates that the direction of the fibers is parallel to the x-axis. For this type of composite, a typical set of reduced stiffnesses is given by

$$\{Q_{ij}\} = \begin{Bmatrix} 25{\cdot}062 & 0{\cdot}25 & 0{\cdot}0 \\ 0{\cdot}25 & 1{\cdot}002 & 0{\cdot}0 \\ 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}5 \end{Bmatrix} \times 10^6\,\text{psi} \tag{54}$$

and

$$Q_{44} = 0{\cdot}2 \times 10^6\,\text{psi}, \quad Q_{55} = 0{\cdot}5 \times 10^6\,\text{psi}, \quad Q_{45} = 0{\cdot}0, \tag{55}$$

In the following table and figures, we use the notations:

$E \equiv$ Extension mode
$B \equiv$ Bending mode
$S =$ Transverse shear mode
$TS =$ Twisting shear mode
$TM =$ Twisting moment mode
$\theta =$ the angle of inclination of the normal to the wave front with the x-axis.

It should be noted that, in general, all these modes are coupled. Thus, an extension mode in fact involves other types of motion. However, from the eigenvectors we are able to tell the dominant deformation in each mode and identify the above designations.

The effect of the amplitude on the shock velocity is summarized in Table 1 for a 0–90-laminate with the shock propagating in the direction $\theta = 15°$. In the table

$$c_T = (G_{LT} + G_{TT})h/2P \tag{56}$$

is the velocity of the shock wave associated with the transverse shear mode. It is obvious that the amplitude of the shock has substantial influence on the transverse shear mode while the other

Table 1. Normal velocities for different shock amplitudes for a 0–90-laminate at $\theta = 15°$

$[\dot{w}]/c_T$ \ c_n/c_T	E	B	TS	TM	S
.0	2.6951	7.9130	2.3711	1.3522	1.0000
.2	2.6973	7.9136	2.3711	1.3527	0.9887
.4	2.7036	7.9154	2.3712	1.3541	0.9546
.6	2.7137	7.9184	2.3713	1.3559	0.8963
.8	2.7273	7.9226	2.3714	1.3578	0.8105
1.0	2.7439	7.9279	2.3715	1.3596	0.6893
1.2	2.7630	7.9345	2.3716	1.3612	0.5112
1.4	2.7839	7.9422	2.3718	1.3626	0.1353

modes are little affected by the amplitude. It is of interest to note that the shock propagation velocity for the shear mode decreases as the amplitude increases. This finding is quite different from that of harmonic wave propagation at large amplitude as observed by Sun and Shafey [6]. In [6], it was found that a larger amplitude resulted in a higher velocity.

The nondimensional shock velocities, c_n/c_T, in the 0–, the 0–90–, the 0–90–0 and the 0–90–90–0 laminates are plotted versus θ in Figs. 1–4, respectively. The strength of the shock is taken as $[w]/c_T = 1$. The linear solutions are also given for comparison. As explained earlier, the bending modes are affected by the geometrical nonlinearity only through the coupling terms B_{ij}. Hence, the velocities for the bending modes for symmetric laminates such as the 0–, 0–90–0– and 0–90–90–0 laminates coincide with the linear solutions.

6. STABILITY OF THE WAVE FRONT

A condition for a wave front to be stable is

$$c_a \leq c_n \leq c_b \tag{57}$$

where c_a and c_b are the velocities of propagation of incremental wave fronts of infinitisimal magnitude immediately ahead and behind the wave front, respectively.

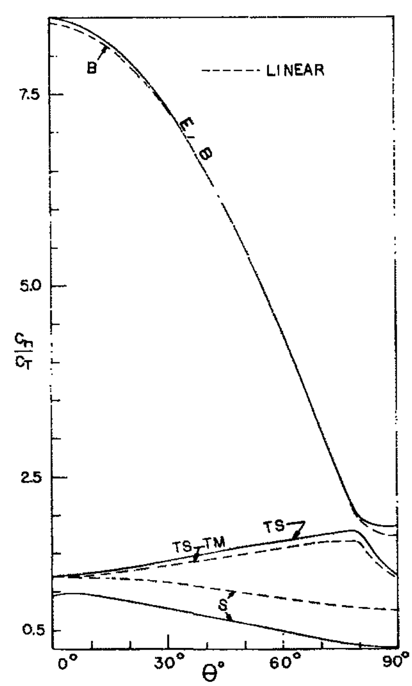

Fig. 1. Normal velocities for the shock waves in a 0-laminate for $[\dot{w}]/c_T = 1$, $c_T = (G_{LT} + G_{TT})h/2P$.

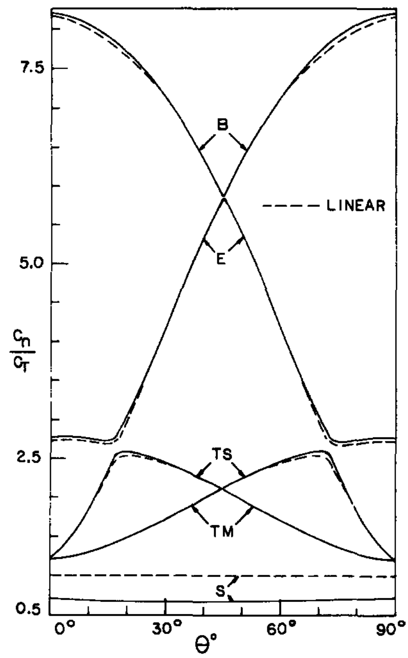

Fig. 2. Normal velocities for the shock waves in a 0–90-laminate for $[\dot{w}]/c_T = 1$. $c_T = (G_{LT} + G_{TT})h/2P$.

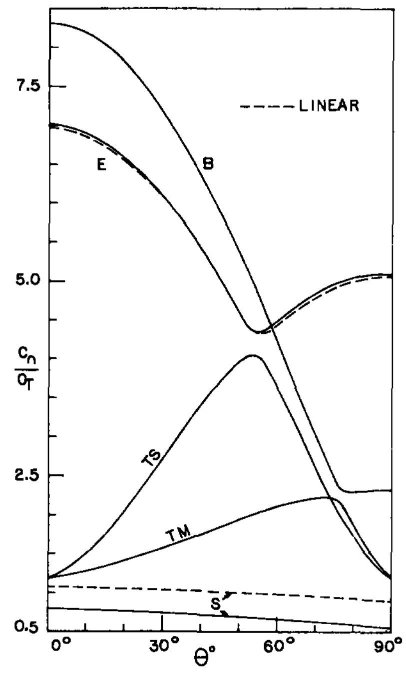

Fig. 3. Normal velocities for the shock waves in a 0–90–0-laminate for $[\dot{w}]/c_T = 1$. $c_T = (G_{LT} + G_{TT})h/2P$.

For a shock wave in initially undisturbed media, the value of c_a is the linear solution of eqn (37) obtained by setting $[\dot{w}] = 0$.

To obtain the incremental shock velocity behind the wave front, we consider an incremental shock of infinitesimal strength for which the propagation velocity satisfies eqn (28) with $[\dot{w}] = 0$ and c_n being replaced by c_b. The main shock front is now considered as an initial disturbance to the trailing incremental shock. Thus the quantities given by eqns (30) and (31) now become

$$\{k^\circ\} = \begin{Bmatrix} \{A^*\} \\ \{B^*\} \end{Bmatrix} \{E_b\} \tag{58}$$

$$\delta_n = \gamma^\circ = \{E_b\}^T\{A^*\}\{E_b\} + \{n\}^T\{T\}^T\{N_b\} \tag{59}$$

where the subscript b is used relative to the main shock.

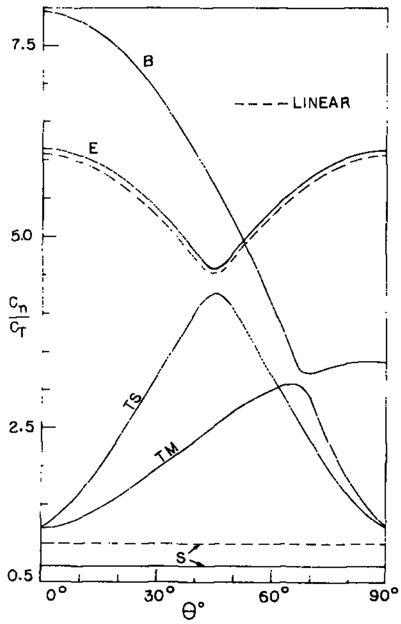

Fig. 4. Normal velocities for the shock waves in a 0–90–90–0-laminate for $[\dot{w}]/c_T = 1$. $c_T = (G_{LT} + G_{TT})h/2P$.

Since the medium is undisturbed ahead of the main shock, we have

$$\frac{\partial w_b}{\partial x} = \left[\frac{\partial w}{\partial x}\right], \frac{\partial w_b}{\partial y} = \left[\frac{\partial w}{\partial y}\right] \tag{60}$$

where $[\partial w/\partial x]$ is the jump in $\partial w/\partial x$ at the main wave front. Using eqns (60) and (4b) we obtain from eqn (27) the relation

$$\{E_b\} = -\frac{[\dot{w}]}{c_n}\{n\} \tag{61}$$

where $[\dot{w}]$ is the strength of the main shock wave. In view of the fact that $\{N_a\} = \{0\}$, we can write

$$\delta_n = \frac{[\dot{w}]^2}{c_n^2}\gamma/c_n^2 + \{[N]\}^T\{T\}\{n\} \tag{62}$$

It can be obtained from eqn (24) that

$$[k_n] = -\frac{[\dot{w}]}{c_n}\{[N]\}^T\{T\}\{n\} \tag{63}$$

From eqns (63) and (20c) we have

$$\{[N]^T\}\{T\}\{n\} = c_n^2 P - k_5. \tag{64}$$

By using the above relations, the equation that c_b has to satisfy assumes the following form

$$\begin{vmatrix} A^*_{11} - c_b^2 P & A^*_{12} & B^*_{11} & B^*_{21} & k_1[\dot{w}] \\ A^*_{21} & A_{22} - c_b^2 P & B^*_{21} & B^*_{22} & k_2[\dot{w}] \\ B^*_{11} & B^*_{12} & D^*_{11} - c_b^2 I & D^*_{12} & k_3[\dot{w}] \\ B^*_{21} & B^*_{21} & D^*_{21} & D^*_{22} - c_b^2 I & k_4[\dot{w}] \\ k_1[\dot{w}] & k_2[\dot{w}] & k_3[\dot{w}] & k_4[\dot{w}] & c_n^2 P(c_n^2 - c_b^2) + \gamma[\dot{w}]^2 \end{vmatrix} = 0. \tag{65}$$

It is obvious that c_b depends on the magnitude of the main shock $[\dot{w}]$ and its normal velocity

c_n. For a particular wave front with given magnitude and velocity, there are five solutions for c_b corresponding to five possible modes of the shock wave. However, the comparison of the velocities must be made between a shock and the similar incremental shock wave. This can be accomplished by comparing the eigenvectors.

For the same numerical example discussed in the previous section, it is found that

$$c_b < c_n < c_a$$

for the transverse shear mode, and

$$c_n > c_b \approx c_a$$

for all other modes. Thus, by the stability criterion given by (57), the transverse shear wave front is unstable while the other shock waves are semi-stable. However, it should be noted that the jump at the shock front in the present analysis represents in an approximate manner the sharp but continuous rise of the particle velocity. The implication of the results of the stability analysis is that if a shock is induced initially by an impulsive load its magnitude would decay in the course of propagation.

Following the similar procedure, stability of the acceleration wave can also be investigated. We obtain

$$c_a = c_n = c_b$$

5. CONCLUSION

Equations governing the shock and acceleration waves with large amplitudes are derived for general laminated plates. The large amplitude is incorporated in the sense of von Karman large deflection theory of plates. Numerical results show that the velocity of the transverse shear shock decreases as the amplitude of the shock increases. It is also found that other types of shock waves are virtually unaffected by the large amplitude. For the acceleration wave, it is found that the velocity does not depend on the strength of the wave front but rather on the initial deformation of the plate.

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REFERENCES

1. F. C. Moon, Wave surface due to impact on anisotropic plates. *J. Composite Materials* **6**, 62 (1972).
2. C. T. Sun, Propagation of shock waves in anisotropic composite plates. *J. Composite Materials* **7**, 366 (1973).
3. D. R. Bland, *Nonlinear Dynamic Elasticity*. Blaisdell, Walthem Mass. (1969).
4. J. E. Ashton and J. M. Whitney, *Theory of Laminated Plates*. Technomic (1970).
5. W. S. Burnside and A. W. Panton, *The Theory of Equations*. Dover, New York (1960).
6. C. T. Sun and N. A. Shafey. Wave propagation in heterogeneous anisotropic plates involving large deflection. *Int. J. Solids Structures* **11**, 99 (1975).

APPENDIX

To obtain the expressions for the nonlinear terms given in eqn (26), we proceed as follows:

Applying operator (25) on $(\partial w/\partial x)^2$, $(\partial w/\partial y)^2$ and $(\partial w/\partial x)(\partial w/\partial y)$, and using the kinematical condition of compatibility, eqn (19), we obtain

$$\left[\left(\frac{\partial w}{\partial x}\right)^2\right] = -\frac{2n_x}{c_n}\frac{\partial w_a}{\partial x}[\dot{w}] + \frac{n_x^2}{c_n^2}[\dot{w}]^2$$

$$\left[\left(\frac{\partial w}{\partial y}\right)^2\right] = -\frac{2n_y}{c_n}\frac{\partial w_a}{\partial y}[\dot{w}] + \frac{n_y^2}{c_n^2}[\dot{w}]^2 \qquad \text{(A-1)}$$

$$\left[\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)\right] = -\frac{[\dot{w}]}{c_n}\left(n_x\frac{\partial w_a}{\partial y} + n_y\frac{\partial w_a}{\partial x}\right) + \frac{n_x n_y}{c_n^2}[\dot{w}]^2.$$

From eqn (17) and the above equations, we obtain

$$\{[N_n]\} = \frac{[\dot{w}]^2}{c_n^2}\{A\}\begin{Bmatrix} \frac{1}{2}n_x^2 \\ \frac{1}{2}n_y^2 \\ n_x n_y \end{Bmatrix} - \frac{[\dot{w}]}{c_n}\{A\}\begin{Bmatrix} n_x \frac{\partial w_a}{\partial x} \\ n_y \frac{\partial w_a}{\partial y} \\ n_x \frac{\partial w_a}{\partial y} + n_y \frac{\partial w_a}{\partial x} \end{Bmatrix}$$

or

$$\{[N_n]\} = \frac{1}{2}\frac{[\dot{w}]^2}{c_n^2}\{A\}\{T\}\{n\} - \frac{[\dot{w}]}{c_n}\{A\}\{T\}\{E_a\}. \tag{A-2}$$

Similarly,

$$\{[M_n]\} = \frac{1}{2}\frac{[\dot{w}]^2}{c_n^2}\{B\}\{T\}\{n\} - \frac{[\dot{w}]}{c_n}\{B\}\{T\}\{E_a\}. \tag{A-3}$$

These are the expressions given by eqn (26). To derive the expression for the nonlinear term $[k_n]$, we apply also the operator given by eqn (25) on $N_x\ (\partial w/\partial x)$, $N_y\ (\partial w/\partial y)$ ... etc. We obtain

$$\left[\frac{\partial w}{\partial x}N_x\right] = -n_x\frac{[\dot{w}]}{c_n}(N_{xa} + [N_x]) + \frac{\partial w_a}{\partial x}[N_x]$$

$$\left[\frac{\partial w}{\partial y}N_{xy}\right] = -n_y\frac{[\dot{w}]}{c_n}(N_{xy_a} + [N_{xy}]) + \frac{\partial w_a}{\partial y}[N_{xy}]. \tag{A-4}$$

Similarly, we can obtain the expressions for $(\partial w/\partial y)\ N_y$ and $(\partial w/\partial x)\ N_{xy}$. Substituting the foregoing results in eqn (24), we obtain

$$[k_n] = -\frac{[\dot{w}]}{c_n}\{n\}^T\{T\}^T\{[N]\} + \{E_a\}^T\{T\}^T\{[N]\} - \frac{[\dot{w}]}{c_n}\{n\}^T\{T\}^T\{N_a\}. \tag{A-5}$$

From eqns (13), (19) and (A-2) we have

$$\{[N]\} = -\frac{1}{c_n}\{A\}\{T\}\begin{Bmatrix}[\dot{u}] \\ [\dot{v}]\end{Bmatrix} - \frac{1}{c_n}\{B\}\{T\}\begin{Bmatrix}[\dot{\psi}_x] \\ [\dot{\psi}_y]\end{Bmatrix}$$
$$+\frac{1}{2}\frac{[\dot{w}]^2}{c_n^2}\{A\}\{T\}\{n\} - \frac{[\dot{w}]}{c_n}\{A\}\{T\}\{E_a\}. \tag{A-6}$$

The expression for $\{[M]\}$ given by eqn (26) can be derived in the same manner. In fact $\{[M]\}$ can be obtained from eqn (A-6) by replacing $\{A\}$ by $\{B\}$ and $\{B\}$ by $\{D\}$. By substituting (A-6) in (A-5), we obtain the third equation in eqn (26).